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String junctions and non-simply connected gauge groups

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Abstract: Relations between the global structure of the gauge group in elliptic F-theory compactifications, fractional null string junctions, and the Mordell-Weil lattice of rational sections are discussed. We extend results in the literature, which pertain primarily to rational elliptic surfaces and obtain $\pi^1(\tilde{G})$ where $\tilde{G}$ is the semi-simple part of the gauge group. We show how to obtain the full global structure of the gauge group, including all U(1) factors. Our methods are not restricted to rational elliptic surfaces. We also consider elliptic K3's and K3-fibered Calabi-Yau three-folds.

Keywords: F-Theory, String Duality, Gauge Symmetry.

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Contents

1. Introduction

The theory of string junctions provides a useful tool for studying aspects of IIB string theory, or equivalently F-theory compactifications on elliptically fibered manifolds X. In this paper we shall use the string junction technology developed in [3]–[7] to elaborate on the work of [1] and [2] concerning the global properties of the gauge group and the Mordell-Weil lattice of the the elliptic fibration.

In particular we will show how to determine the global structure of the gauge group in F-theory compactifications on elliptic K3 manifolds, and propose a method to obtain the global structure for K3-fibered Calabi-Yau three-folds. The gauge groups which can arise in this context are subgroups of a single infinite dimensional Lie group [5, 6], which is a broken symmetry of the F-theory compactification. This infinite dimensional group is simply connected, having a weight lattice equal to the root lattice. We shall show that information about the global structure of the subgroups is simply encoded in the lattice of null string junctions.

Locally, the gauge group G is a product of simple Lie groups and some U(1) factors. We will write the semi-simple part of the gauge group (containing no U(1) factors) as $\tilde{G}$. The discussion of the global structure of the gauge group in [1] pertains to $\pi^1(\tilde{G})$. The authors of [1] considered compactification on an elliptic K3 in the limit of a stable degeneration to a pair of intersecting rational elliptic surfaces R_1 and R_2. In this limit, the states carrying gauge charges correspond to certain elements of $H_2(\mathcal{R}_i, Z)$, which may be represented by string junctions in the base of $\mathcal{R}_i$. The map from the elements of $H_2(\mathcal{R}_i, Z)$ to the weight lattice was found to have a non-trivial cokernal. The missing representations of the algebra were found to imply the gauge group $\tilde{G} = \tilde{G}_1 \times \tilde{G}_2$, with $\pi^1(\tilde{G}_i) = T(\Phi_i)$, where $T(\Phi)$ is the torsion part of the Mordell-Weil lattice of $\mathcal{R}_i$. The Mordell-Weil lattice Φ (see [8]) is the lattice of rational sections of the elliptic surface, which are closely related to the poles of the Seiberg-Witten differential [9, 10]. $T(\Phi)$ consists of sections S with the property that $nS = 0$, the zero section, for some integer n.

The Mordell-Weil lattice has been completely tabulated for all rational elliptic surfaces [11]. In [2] it was observed by direct comparison that $T(\Phi)$ for a rational elliptic surface is identical to a lattice generated by "improper" null junctions on the P^1 base. These null junctions carry fractional (p,q) charges, and are not realizable as membranes upon lifting to F-theory. Their relation to geometrical objects, i.e. sections of the rational elliptic surface, is not at all manifest. By combining the observations of [2] with the results of [1], one concludes that these null junctions must somehow generate $\pi^1(\tilde{G}_i)$.

We will give a direct argument showing why these null junctions generate $\pi^1(\tilde{G}_i)$. To do so we will make use of an isomorphism between the center of the universal cover of $\tilde{G}$ and the quotient of the string junction lattice, which includes "improper" fractional junctions, by the lattice of proper junctions. There is a map from the fractional null junctions to the subgroup of the center of the universal cover which acts trivially on all states in the spectrum. This map gives not just $\pi^1(\tilde{G})$, but the complete global structure of $\tilde{G}$. Furthermore, this map is not be restricted to rational elliptic surfaces. We shall also consider elliptic K3's and K3 fibered Calabi-Yau three-folds, taking into account states which decouple in the limit in which a K3 becomes a pair of intersecting rational elliptic surfaces. These states are missing in the analysis of [1], as they correspond to infinitely massive strings stretching between the P^1 base of each rational elliptic surface. In the dual heterotic description on T^2, these states are strings wrapping cycles of the T^2, the radii of which go to infinity as the K3 degenerates to a pair of rational elliptic surfaces. The presence of these states, which do not decouple for a generic K3, modifies the global structure of the gauge group computed in [1]. This modification can be determined from the lattice of null junctions in the base of the K3.

The null junctions give the global structure of the semi-simple part of the gauge group, $\tilde{G}$. One can easily extend the analysis to compute the full global structure including all the U(1) components. Generically, the U(1) components of the gauge group do not enter as global factors. The methods we use to compute $\pi^1(\tilde{G})$ can be easily extended to compute the correlation between U(1) charges and charges under the center of $\tilde{G}$.

Some new subtleties appear when analyzing the spectrum of F-theory compactified on elliptically fibered Calabi-Yau three-folds. These subtleties arise due to singularities of the discriminant curve. The gauge group and hypermultiplet spectrum of such compactifications have been studied by a variety of authors [17, 18, 19, 21, 22, 1]. The simplest case to analyze involves K3 fibered Calabi-Yau three-folds, taking again the limit in which generic K3 fibers become pairs of intersecting rational elliptic surfaces. In [1], it was suggested that the relation between the torsion component of the Mordell-Weil lattice and $\pi^1(\tilde{G})$ might persist for such a Calabi-Yau. No attempt has yet been made to address this question in general. The Mordell-Weil lattice is more difficult to obtain on an elliptic three-fold. Furthermore, unlike the rational elliptic surface, there is no known theorem relating $\pi^1(\tilde{G})$ to the torsion component of the Mordell-Weil lattice, although in some instances (see [1]) this relation still holds. Nonetheless, we shall argue that one can still use string junction technology to compute the global properties of the gauge group. It is possible to define a string junction lattice on an elliptically fibered Calabi-Yau three-fold (see [16]), and to associate hypermultiplets with elements of this lattice. Unlike a K3, some elements of the string junction lattice of the three-fold X are not contained in $H_2(X, Z)$. Currently, we do not know if $\pi^1(\tilde{G})$ is isomorphic to $T(\Phi)$ in all instances.

The organization of this paper is as follows. Section 2 is a brief review of string junction lattices in the context of F-theory on an elliptic surface. In section 3, we show why fractional null junctions generate $\pi^1(\tilde{G})$. In section 4, we show how to compute the global structure of the gauge group including all the U(1) factors. In section 5, we review some facts about gauge groups and hypermultiplets in elliptically fibered Calabi-Yau three-folds, and propose a method to obtain the global structure of the gauge group using string junctions.

2. Review of string junctions

The string junction lattice associated with F-theory compactified on an elliptic surface was defined and studied in [3]–[7]. We give a very brief review of this work below. The reader familiar with this technology may wish to skip to the next section.

F-theory on an elliptic surface X describes IIB string theory on P^1, with 7-branes at points in the P^1. In the absence of other branes (such as a three-brane) the string junction lattice consists of the equivalence classes of string junctions having endpoints on the 7-branes. In the F-theory description, an individual 7-brane corresponds to a point in the base (a discriminant locus) over which there is a degenerate elliptic fiber of Kodaira type I_1 [23], having a unique vanishing cycle. To define the junction lattice, one initially considers all seven branes to be separated in P^1. An example of a string junction is illustrated in figure 1. Each segment of the junction is an oriented string with charges (p, q), which are conserved at branching points. These charges

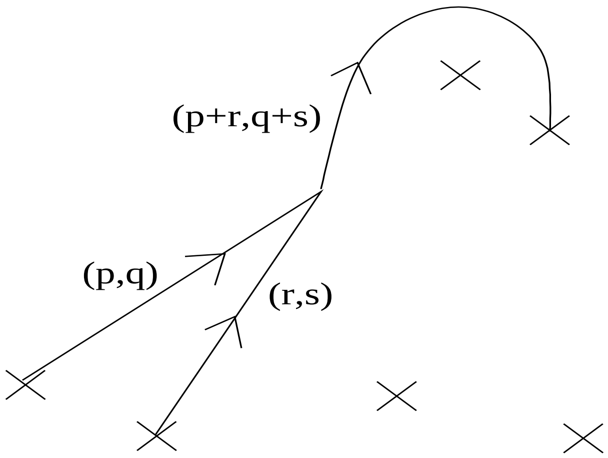

Figure 1: An example of a string junctions in P^1. The crosses indicate 7-branes.

correspond to one cycles in the elliptic fiber over each segment, and the the strings junctions lift to membranes in F-theory.

Individual 7-branes, labelled by an index i, carry relatively prime charges (p_i, q_i) indicating the vanishing cycle of the i'th I_1 fiber. A string ending on a 7-brane must carry the same charges. Unless a three-brane probe is present, the membrane associated with a string junction does not have a boundary, and represents an element of $H_2(X, Z)$.

The vanishing cycle of the i'th I_1 fiber is the eigenvector of the monodromy matrix K_i associated with that fiber;

$$K_i = \begin{pmatrix} 1 + p_i q_i & -p_i^2 \\ q_i^2 & 1 - p_i q_i \end{pmatrix} . \tag{2.1}$$

To determine the equivalence classes of string junctions on P^1, it is convenient to move the 7-branes onto the real axis with branch cuts extending vertically downward. A segment of a string junction having charge (p, q) on the left side of the branch cut of the i'th 7-brane has charge $K_i(p, q)$ on the right side of the branch cut. Such a segment can be pulled across the 7-brane so that it no longer crosses the branch cut. However, because of an effect dual to a Hanany-Witten transition [12], an additional segment appears stretching between the original segment and the 7-brane, as illustrated in figure 2. The charges of the new segment are determined by charge conservation to be $(p', q') = K_i(p, q) - (p, q)$, which is proportional to the vanishing cycle (p_i, q_i). The two junctions related by pulling this segment across the 7-brane are said to be in the same equivalence class. Thus, the equivalence classes of junctions can be determined by considering only junctions which lie entirely above the real axis, as in figure 2. The equivalence classes are then labelled by vectors $\vec{J} = (N_1, N_2, \ldots, N_n)$, where the integers N_i indicate the number and orientation of strings ending on the i'th 7-brane. The charges of the string segment ending on the i'th 7-brane are

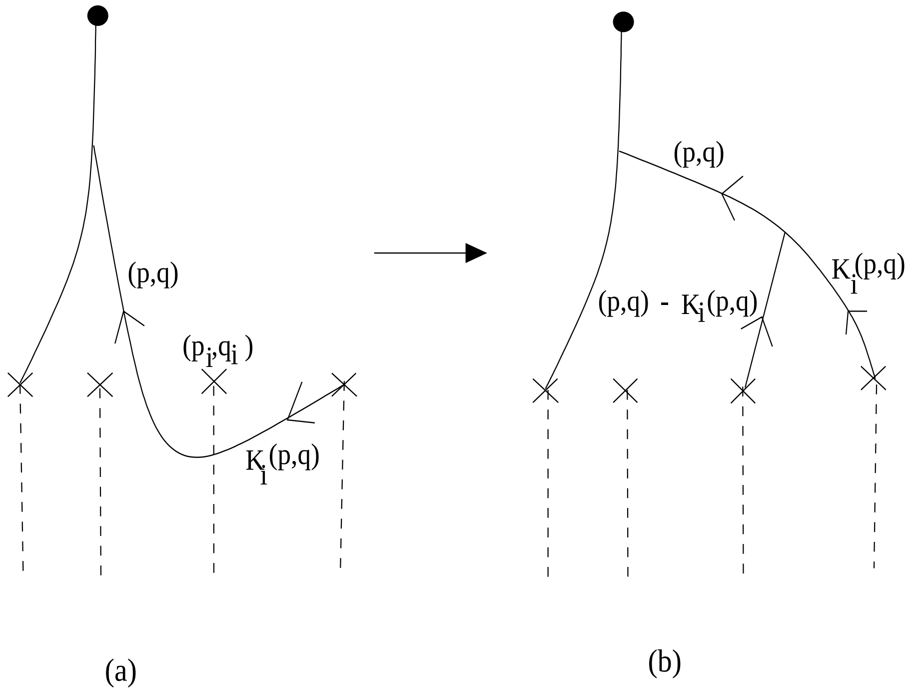

Figure 2: Two equivalent string junctions related by a Hanany-Witten transition. Figure 2 is a canonical representation of the equivalence class, in which the junction lies entirely above the real axis. The dashed lines are branch cuts going down from each 7-brane.

$N_i(p_i, q_i)$. For junctions which lift to a membrane without a boundary, the total (p, q) charge vanishes; $\sum_i N_i(p_i, q_i) = (0, 0)$. For the moment, we will not make this restriction and consider membranes with a (p, q) cycle as a boundary. Later, to obtain states in F-theory in the absence of three-branes, we will glue such membranes together to get one without a boundary.

The basis in which an equivalence class $\vec{J}$ is labelled by the integers N_i is not the most useful one. By allowing certain unphysical junctions with fractional N_i charges, one can find another basis in which there is a a manifest correspondence between elements of the junction lattice and weights of a Lie Algebra. The physical junctions then span a sub-lattice of the full string junction lattice. Consider the sub-lattice of junctions which end on a certain subset of the 7-branes, and which have a total asymptotic charge $(p, q) = \sum_i N_i(p_i, q_i)$. Let us assume that this subset of 7-branes is collapsible. In the F-theory description, this means that the corresponding I_1 fibers can collide to form a more singular Kodaira fiber. The more singular Kodaira fibers have an A-D-E classification, indicating the enhanced gauge symmetry algebra of the configuration of coincident 7-branes. The junction lattice is initially defined before collapsing the 7-branes. In order to map elements of the junction lattice to weights of the algebra, one requires a quadratic form on the junction lattice. This form was defined in [3] and, for a $K3$, coincides with the intersection form on $H_2(X, Z)$ when considering junctions with vanishing (p, q). Linearly independent junctions $\vec{\alpha}_k$ with vanishing (p, q) and self-intersection number -2 are associated with the simple roots of the (simply laced) Lie algebra. The self intersection matrix of these junctions is minus the Cartan matrix; $(\vec{\alpha}_k, \vec{\alpha}_l) = -C_{kl}$. When 7-branes collide, these junctions have zero length, giving rise to massless vector multiplets and an enhanced gauge symmetry. As a simple example, figure 3 illustrates the simple root junction for two I_1 fibers which collide to give an I_2 (or A_1) singularity. In this case, the resulting gauge symmetry algebra is that of SU(2).

One can define a set of fundamental weight junctions $\vec{w}_i$ with vanishing (p, q) satisfying $(\vec{w}^i, \vec{\alpha}_j) = -\delta^j_i$. These junctions are generically "improper" meaning that may have fractional N_i charges. To get a complete basis, one defines another pair of junctions $\vec{w}_p$ and $\vec{w}_q$, known as "extended weights," which are orthogonal to the weight junctions $\vec{w}_i$ and which carry charges $(p, q) = (1, 0)$ and $(0, 1)$ respectively. The junctions $\vec{w}_p$ and $\vec{w}_q$ are also generically improper. Any proper junction $\vec{J}$ can be written as a linear combination of the improper basis junctions.

$$\vec{J} = p\vec{w}_p + q\vec{w}_q + a^i\vec{w}_i \,, \tag{2.2}$$

where the integers a^i are the Dynkin labels. The integer asymptotic charges p and q are related to U(1) charges and are called "extended Dynkin labels".

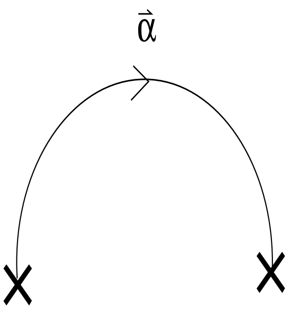

Figure 3: The simple root junction of an I_2 Kodaira fiber, which has been split into two I_1 fibers, corresponding to 7-branes with the same charge. The associated algebra is SU(2).

Multiplets under various representations of the algebra may be found by identifying highest weights, however it will not be necessary for us to do this, since we are seeking only the global structure of the gauge group.

For F-theory on an elliptic surface, or IIB theory on P^1, there will generically be several sets of coincident 7-branes labelled by an index m. These coincident sets give rise to a gauge group which is locally the product of A-D-E groups for each m and some U(1) factors. We will refer to the semi-simple part of the gauge group as $\tilde{G}$, and the entire group, including the U(1) factors, as G. A general string junction can be written as

$$\vec{J} = \sum_m p_m\vec{w}_{p,m} + q_m\vec{w}_{q,m} + a^i_m\vec{w}_{i,m} \,. \tag{2.3}$$

Since we assume the absence of three-branes, all strings must end on 7-branes and the total asymptotic charge vanishes,

$$\sum_m p_m = \sum_m q_m = 0 \,. \tag{2.4}$$

Here again p_m, q_m, a^i_m are taken to be integers. For a physical (or proper) string junction, the charges N_i must also be integer. Combining this constraint with the vanishing of the total asymptotic charge restricts the representations which can ap-

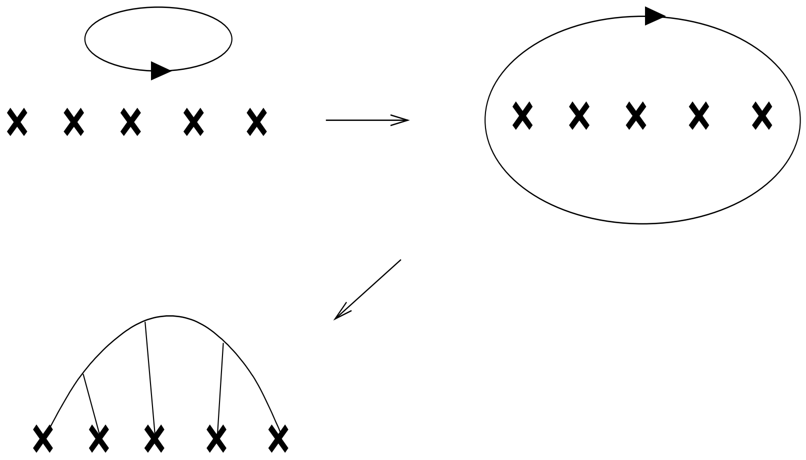

Figure 4: A proper null junction, which is related by Hanany Witten transitions to a contractable closed string loop. The loop surrounds all the 7-branes, around which the total $\mathrm{SL}(2,\mathbb{Z})$ monodromy is 1.

pear in the spectrum. We will see later that this restriction is consistent with a particular non-simply connected gauge group.

Note that one could consider all the 7-branes collectively, instead of grouping them according to the unbroken gauge symmetry, and write

$$\vec{J} = \sum_i a_i \vec{w}_i \,. \tag{2.5}$$

Here one should view the $\vec{w}_i$ as the weights of an infinite dimensional Lie algebra [5, 6], of which only a finite dimensional sub-algebra is realized as a gauge symmetry. In this case the weight lattice is the same as the root lattice, and the weights $\vec{w}_i$ are proper. It follows that there is no constraint on the representations which may appear, and the infinite dimensional Lie group is simply connected.

A subset of the the integers p_m and q_m are charges for the U(1) factors in the gauge symmetry group. There are actually four fewer conserved U(1) charges than the number of indices m would indicate. Two of the charges are not independent, due to (2.4). Two more of the charges are not conserved, due to an equivalence relation in which one adds "proper null junctions" to $\vec{J}$.

The null junctions are of particular interest in this paper. These junctions have vanishing intersection with any other junction. The *proper* null junctions are topologically trivial, meaning that they are related by Hanany-Witten transitions to a contractable closed string loop. If the base manifold on which the closed string junctions live is a P^1 (as for an elliptic K3, or a rational elliptic surface), then any contractable closed string loop can be viewed as a loop surrounding all of the 7-branes, as in figure 4. The total monodromy upon encircling all the 7-branes is the identity. There are 12 7-branes in the base of a rational elliptic surface, and 24 7-branes in the base of an elliptic K3. Via Hanany-Witten transitions, such a loop is equivalent to a junction lying above the real axis, with endpoints on the 7-branes. This junction carries no symmetry charges, and thus has the form

$$\vec{N} = \sum_m (P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}) \tag{2.6}$$

with $\sum_m P_m = \sum_m Q_m = 0$. The (p, q) charge on the upper part of a string loop surrounding all the 7-branes is arbitrary, so the lattice of null junctions is two dimensional.

We shall also have reason to consider "improper null junctions" which are fractions of the proper null junctions, with integer P_m and Q_m but fractional N_i charges. We will eventually find that the quotient of the lattice of all null junctions (proper and improper) by the lattice of proper null junctions is isomorphic to $\pi^1(\tilde{G})$, where $\tilde{G}$ is the semi-simple part of the gauge group.

3. Non-simply connected gauge groups for elliptic surfaces

The gauge symmetry algebra of F-theory compactified on $K3$ can be read directly from the A-D-E type of the Kodaira fibers in the K3. However determining the gauge group G takes more work. This problem was discussed in [1], where $\pi^1(\tilde{G})$ was obtained by studying the spectrum for compactifications on K3's which have degenerated to pairs of intersecting rational elliptic surfaces. We shall address this problem using a method which is not restricted to rational elliptic surfaces, and which yields the full global structure of the gauge group. In doing so, we will explain the observations of [2] relating improper null junctions to the torsion component of the Mordell-Weil lattice of a rational elliptic surface.

If $\pi^1(G)$ is non-trivial, hypermultiplets in certain representations of the algebra can not appear in the spectrum. States charged under G are realized as string junctions with vanishing asymptotic (p, q). The constraint of vanishing (p, q) together with the requirement that physical string junctions have integer charges is sufficient to determine the global structure of the gauge group.

Consider an elliptic fibration over P^1, with Kodaira fibers K_m (coincident 7-branes) at points m in P^1. The gauge symmetry algebra $\mathcal{G}$ is the direct sum of the A-D-E algebras of each Kodaira fiber, $\mathcal{G} = \sum_m \mathcal{G}_m$. There is a sub-lattice of string junctions associated with each m, comprised of junctions having endpoints only at the point m. A junction associated with the m'th point may be written as

$$\vec{J}_m = p_m \vec{w}_{p,m} + q_m \vec{w}_{q,m} + a_{i,m} \vec{w}_{i,m} \,. \tag{3.1}$$

To obtain physical states (in the absence of three-branes) these junctions must be combined to give one without any asymptotic charge;

$$\vec{J} = \sum_m \vec{J}_m \,, \tag{3.2}$$

with $\sum_m p_m = \sum_m q_m = 0$. One must also impose the constraint that each $\vec{J}_m$ is a proper junction. Of course not all such states will be BPS, or stable, but must nevertheless be considered to obtain the global properties of the gauge group.

Each $\vec{J}_m$ belongs to some representation R_m[1] of the universal covering group G_m associated with the algebra $\mathcal{G}_m$. G_m has a center C_m. We will shortly see that when $\vec{J}_m$ is a proper junction, the action of the center C_m in the representation R_m is determined entirely by the asymptotic charges p_m and q_m. In the absence of constraints on p_m and q_m, the gauge group (up to U(1) factors) would be a direct product of the universal covers G_m. However there are elements in the center of $\prod G_m$ which act trivially in all representations containing proper junctions with $\sum_m p_m = \sum_m q_m = 0$. This is what gives rise to a non-simply connected gauge group.

Before making the above statements more precise, we shall need some simple facts about group theory. The center C of the universal cover of a simple Lie group has the structure $\prod_l Z_l$, and a corresponding lattice $\mathcal{L}_C$. $\mathcal{L}_C$ is given by the quotient of the weight lattice by the root lattice,

$$\mathcal{L}_C = \mathcal{L}_{\vec{w}}/\mathcal{L}_{\vec{\alpha}}\,. \tag{3.3}$$

This quotient gives a natural map from weights to the center lattice. This map determines how a particular representation transforms under the center. For instance for SU(2) $\mathcal{L}_C$ is a Z_2 lattice comprised of 0 and 1 (corresponding to center elements 1 and -1 of SU(2)). The fundamental weight $\vec{w}$ maps to 1. A generic weight $\vec{W} = n\vec{w}$ maps to n *mod* 2. Integer spin representations, with weights of even n, transform trivially under the center, while half integer spin representations, with odd n, do not. In general the root vectors, or weights in the adjoint representation, map to the trivial element of the center.

More generally, suppose a weight vector maps to an element c in the center lattice $\mathcal{L}_C$. Let Ψ be an element of a representation containing that weight vector. Then under the action of an element of the center represented by c' in $\mathcal{L}_C$, Ψ transforms as follows

$$\Psi \to \exp(i\mathbf{c}'\cdot\mathbf{c})\Psi\,, \tag{3.4}$$

where the dot product is taken with respect to an appropriate metric on $\mathcal{L}_C$. For SU(2), where $\mathbf{c}$ and $\mathbf{c}'$ take values 0 or 1 (mod 2), the above phase factor is $\exp(i\pi\mathbf{c}\mathbf{c}')$.

3.1 Junctions and representations of the center; an I_3 example

For the moment, let us just consider string junctions with endpoints on a single set of coincident 7-branes, with no requirement that the asymptotic p and q vanish. For the sake of clarity, we will work in the context of a particular example in which $\vec{J}$ are

[1] The exact representation can not be determined without knowing the highest weight. However this is irrelevant to our purposes since the behavior under the center of the universal cover does not depend on the highest weight.

junctions ending on the locus of an I_3 fiber, which is associated with an SU(3) algebra. The I_3 fiber may be split into three I_1 fibers, each with vanishing cycle $[p,q]=[1,0]$. Equivalence classes of junctions are labelled by three charges, $\vec{J}=(N_1,N_2,N_3)$. In this basis, the intersection form is $g_{ij}=-\delta_{ij}$. The simple root junctions are

$$\begin{aligned}\vec{\alpha}_1 &= (1,-1,0)\\ \vec{\alpha}_2 &= (0,1,-1)\,.\end{aligned} \tag{3.5}$$

The fundamental weight junctions are

$$\begin{aligned}\vec{w}_1 &= (2/3,-1/3.-1/3)\\ \vec{w}_2 &= (1/3,1/3,-2/3)\,.\end{aligned} \tag{3.6}$$

There is a single extended weight in this case, given by

$$\vec{w}_p = (1/3,1/3,1/3)\,. \tag{3.7}$$

We define the full junction lattice to consist of all junctions of the form $\vec{J}=p\vec{w}_p+a_1\vec{w}_1+a_2\vec{w}_2$ with integer p,a_1 and a_2. In terms of the N_i charges, one has

$$\begin{aligned}p &= N_1+N_2+N_3\\ a_1 &= N_1-N_2\\ a_2 &= N_2-N_3\,.\end{aligned} \tag{3.8}$$

The full junction lattice contains a sub-lattice of proper junctions with integer N_i.

Let us restrict ourselves for the moment to the case $p=0$. For $p=0$, the quotient of the string junction lattice by the lattice of proper junctions is the same as the quotient of the weight lattice by the root lattice. In this case, this quotient gives $\mathbb{Z}_3$. We will define a linear map, M_1, which maps weight vectors to the $\mathbb{Z}_3$ lattice according to this quotient;

$$\begin{aligned}M_1(\vec{w}_1) &= 2 \bmod 3\\ M_1(\vec{w}_2) &= 1 \bmod 3\,.\end{aligned} \tag{3.9}$$

Strictly speaking, the quotient gives a map is defined only up the automorphism of $\mathbb{Z}_3$ which exchanges 2 mod 3 with 1 mod 3. In (3.9) we have made a particular choice. For $p=0$, the proper junctions are in the root lattice and are mapped to zero by M_1. We extend the action of the map M_1 to junctions with non-vanishing p as follows;

$$M_1(\vec{w}_p) = 0 \bmod 3\,. \tag{3.10}$$

The map M_1 determines the behavior of junctions under the center of SU(3). Let us represent an element of the center by an element $\mathbf{c}$ in the $\mathbb{Z}_3$ lattice, the action of this element of the center on a component $\Psi_{\vec{J}}$ of a representation containing $\vec{J}$ is given by

$$\Psi_{\vec{J}} \to \exp\left(i\mathbf{c}\cdot M_1(\vec{J})\right)\Psi_{\vec{J}} = \exp\left(\frac{2\pi i}{3}\mathbf{c}M_1(\vec{J})\right)\Psi_{\vec{J}}\,. \tag{3.11}$$

Note that we have made a choice of metric with which to take the dot product. This metric is well defined only up to an automorphism of $\mathbb{Z}_3$. The element of the center which $\mathbf{c}$ represents depends on this choice.

We now observe that any combination of weight vectors $a_i\vec{w}_i$, can be made into a proper junction by adding integer multiples of $\vec{w}_p$. For instance, $\vec{w}_p+\vec{w}_1$ is a proper junction, whereas $\vec{w}_1$ is not. This permits us to define another linear map M_2 from junctions to the center lattice, such that M_2 maps all the proper junctions to zero. M_2 acts on the weight vectors the same way as M_1;

$$\begin{aligned} M_2(\vec{w}_1) &= M_1(\vec{w}_1) = 2 \bmod 3 \\ M_2(\vec{w}_2) &= M_1(\vec{w}_2) = 1 \bmod 3\,. \end{aligned} \tag{3.12}$$

but acts non-trivially on $\vec{w}_p$;

$$M_2(\vec{w}_p) = 1 \bmod 3\,. \tag{3.13}$$

One can easily verify that if a junction $\vec{J}_{\mathcal{P}}$ is proper, then

$$M_2(\vec{J}_{\mathcal{P}}) = 0\,. \tag{3.14}$$

Due to (3.14), the behavior of any proper junction under the center, which depends on its map under M_1, is determined entirely by its asymptotic charge p;

$$\begin{aligned} M_1(\vec{J}_{\mathcal{P}}) = M_1(p\vec{w}_p + a_1\vec{w}_1 + a_2\vec{w}_2) &= \\ M_1(a_1\vec{w}_1 + a_2\vec{w}_2) = M_2(a_1\vec{w}_1 + a_2\vec{w}_2) &= \\ -M_2(p\vec{w}_p) = -p \bmod 3\,. \end{aligned} \tag{3.15}$$

It will also prove useful to define linear maps $\tilde{M}_1$ and $\tilde{M}_2$ of junctions to the cover $\mathbb{Z}$ of $\mathbb{Z}_3$;

$$\begin{aligned} \tilde{M}_1(\vec{w}_1) &= 2 \\ \tilde{M}_1(\vec{w}_2) &= 1 \\ \tilde{M}_1(\vec{w}_p) &= 0 \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \tilde{M}_2(\vec{w}_1) &= 2 \\ \tilde{M}_2(\vec{w}_2) &= 1 \\ \tilde{M}_2(\vec{w}_p) &= 1\,. \end{aligned} \tag{3.17}$$

JHEP07(2001)002

These maps have the property that division is well defined, in the sense that

$$\frac{1}{l}\tilde{M}\left(\vec{J}\right) = \tilde{M}\left(\frac{\vec{J}}{l}\right) \tag{3.18}$$

whenever l is integer and $\vec{J}/l$ is a junction with integer p, q and a_i. We shall eventually use null junctions obtained from fractions of proper null junctions to represent the elements in the center of the universal cover of $\tilde{G}$ which act trivially on all states in the spectrum.

3.2 Junctions and representations of the center for a general A-D-E singularity

In general, the following is true for any collection of 7-branes associated with an A-D-E algebra.

The quotient of the lattice of junctions $\mathcal{J}$ by the lattice of proper junctions $\mathcal{J}_{\mathcal{P}}$ is the lattice $\mathcal{L}_C$ associated with the center of the universal cover of $\tilde{G}$

$$\mathcal{L}_C = \mathcal{J}/\mathcal{J}_{\mathcal{P}}\,. \tag{3.19}$$

There exist two linear maps from the junction lattice $\mathcal{J}$ to the center lattice $\mathcal{L}_C$, M_1 and M_2, with the following properties. M_2 maps junctions by quotienting by the proper junctions,

$$M_2(\vec{J}) = \vec{J} \bmod \mathcal{J}_{\mathcal{P}}\,, \tag{3.20}$$

while M_1 maps junctions to the center lattice according to

$$M_1(\vec{J}) = M_1(p\vec{\omega}_p + q\vec{\omega}_q + a_i\vec{\omega}_i) = M_2(a_i\vec{\omega}_i)\,. \tag{3.21}$$

The behavior of junctions under the center of the universal cover is determined by the map M_1. The action of an an element of the center represented by $\mathbf{c} \in \mathcal{L}_C$ in a representation containing the junction $\vec{J}$ is given by

$$\Psi_{\vec{J}} \to \exp(i\mathbf{c}\cdot M_1(\vec{J}))\Psi_{\vec{J}}\,, \tag{3.22}$$

where the dot product is with respect to an appropriate metric on the center lattice. The element of the center which $\mathbf{c}$ represents is fixed only after choosing the maps M_1, M_2 and the metric, both of which are defined only up to an automorphism of the center.

Since M_2 maps proper junctions $\vec{J}_{\mathcal{P}}$ to zero, the action of M_1 on proper junctions is determined entirely by the asymptotic charges p and q

$$M_1(\vec{J}_{\mathcal{P}}) = M_1(p\vec{\omega}_p + q\vec{\omega}_q + a_i\vec{\omega}_i) = M_2(a_i\vec{\omega}_i) = -M_2(p\vec{\omega}_p + q\vec{\omega}_q)\,. \tag{3.23}$$

For the A-D-E groups, the center of the universal cover is either of the form $\mathbb{Z}_N \times \mathbb{Z}_M$ or $\mathbb{Z}_N$, which is consistent with the fact that there at most two asymptotic charges,

Kodaira - ADE type	universal cover	center	decomposition
$III - A_1$	SU(2)	$\mathbb{Z}_2$	AAC
$IV - A_2$	SU(3)	$\mathbb{Z}_3$	$AAAC$
$I^*_{2n} - D_{4+2n}$	Spin$(8+4n)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$A^{4+2n}BC$
$I_{2n+1} - D_{4+2n+1}$	Spin$(10+4n)$	$\mathbb{Z}_4$	$A^{5+2n}BC$
$IV^* - E_6$	E_6	$\mathbb{Z}_3$	$AAAAABCC$
$III^* - E_7$	E_7	$\mathbb{Z}_2$	$AAAAAABCC$
$II^* - E_8$	E_8	I	$AAAAAAABCC$
$I_n - A_{n-1}$	SU(n)	$\mathbb{Z}_n$	A^n

Table 1: We list the Kodaira/ADE singularities, the center of the universal cover, and a decomposition into I_1 fibers (or 7-branes) with monodromy matrices of type A, B or C arranged from left to right on the real axis. These matrices are (up to an overall SL$(2,\mathbb{Z})$ conjugation), $A = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 5 & -16 \\ 1 & -3 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 3 & -4 \\ 1 & -1 \end{smallmatrix}\right)$.

fiber	$\vec{w}_p$	$\vec{w}_q$
III	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2}, -1)$
IV	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$
I^*_0	$(0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2})$
IV^*	$(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3})$	$(1, 1, 1, 1, 1, -3, -1, -1)$
III^*	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 2, 1, 1)$	$(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -5, -2, -2)$
II^*	$(-1, -1, -1, -1, -1, -1, -1, 4, 2, 2)$	$(3, 3, 3, 3, 3, 3, 3, -11, -5, -5)$
I_n	$(\frac{1}{n}, \frac{1}{n}, \cdots)$	

Table 2: This table lists some Kodaira fibers and the extended weights $\vec{w}_p$ and $\vec{w}_q$ written in the N_i basis.

p and q. One can also define maps $\tilde{M}_1$ and $\tilde{M}_2$ of junctions to $\mathbb{Z} \times \mathbb{Z}$ or just $\mathbb{Z}$ as in (3.17).

Note that the center of the universal cover is isomorphic to the quotient of the lattice of junctions of the form $P\vec{w}_p + Q\vec{w}_q$ by its proper sub-lattice. The reader may explicitly verify this by inspecting table 1(a) and table 1(b). We will later find it useful to represent elements in the center of the universal cover $\mathbf{c}$ by junctions of this form;

$$\mathbf{c} = M_2(\vec{J}_c) = M_2(P\vec{w}_p + Q\vec{w}_q)\,. \tag{3.24}$$

The action of $\mathbf{c}$ in a representation containing proper junctions $\vec{J}_{\mathcal{P}} = p\vec{w}_p + q\vec{w}_q + \cdots$ can be written as

$$\Psi_{\vec{J}_{\mathcal{P}}} \to \exp(-i\tilde{M}_2(P\vec{w}_p + Q\vec{w}_q) \cdot \tilde{M}_2(p\vec{w}_p + q\vec{w}_q))\Psi_{\vec{J}_{\mathcal{P}}}\,. \tag{3.25}$$

3.3 $\pi^1(G)$ for elliptic surfaces.

To construct physical string junctions arising for compactifications of F-theory on an elliptic surface, one must glue together junctions with endpoints on different collections of coincident 7-branes so that the total (p, q) charge vanishes. Junctions then have the form

$$\vec{J} = \sum_m \vec{J}_m = \sum_m (p_m \vec{w}_{p,m} + q_m \vec{w}_{q,m} + a_{i,m} \vec{w}_{i,m}) \,, \tag{3.26}$$

with $\sum_m p_m = \sum_m q_m = 0$. The universal cover of the semi-simple part of the gauge group is given by $\prod_m G_m$, with center

$$\mathcal{C} = \prod_m \mathcal{C}_m \,. \tag{3.27}$$

We denote an element in the lattice $\mathcal{L}_C$ associated with $\mathcal{C}$ by $\mathbf{c}$, and an element in the sub-lattice associated with $\mathcal{C}_m$ by $\mathbf{c}_m$. In general $\mathcal{C}_m$ is a lattice of the form $\mathbb{Z}_{(N_m)} \times \mathbb{Z}_{(M_m)}$ or $\mathbb{Z}_{(N_m)}$. The maps M_1 and M_2 defined in the previous section are now defined for each index m, giving maps $\vec{J} \to \mathcal{L}_C$. The action of an element of $\mathcal{C}$ in a representation containing the junction $\vec{J}$ is given by

$$\Psi_{\vec{J}} \to \exp(i\mathbf{c} \cdot M_1(\vec{J}))\Psi_{\vec{J}} = \exp\left(i \sum_m \mathbf{c}_m \cdot M_1(\vec{J}_m)\right) \Psi_{\vec{J}} \,, \tag{3.28}$$

where again the dot product is defined with respect to an appropriate metric on $\mathcal{L}_{\mathcal{C}}$.

Physical states correspond to proper junctions $\vec{J}_{\mathcal{P}}$ with $\sum_m p_m = \sum_m q_m = 0$. The global structure of the gauge group is determined by finding the elements in $\mathcal{C}$ which act trivially on all such junctions. In other words, we seek $\mathbf{c}$ such that

$$\exp(i\mathbf{c} \cdot M_1(\vec{J}_{\mathcal{P}})) = 1 \,. \tag{3.29}$$

$\mathbf{c}$ may be represented by a junction $\vec{J}_c$ of the form

$$\vec{J}_c = \sum_m \vec{J}_{c,m} = \sum_m P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m} \,, \tag{3.30}$$

for which

$$\mathbf{c} = M_2(\vec{J}_c) \,. \tag{3.31}$$

Thus we seek $\vec{J}_c$ such that

$$\exp(iM_2(\vec{J}_c) \cdot M_1(\vec{J}_{\mathcal{P}})) = 1 \tag{3.32}$$

for all $\vec{J}_{\mathcal{P}}$.

A natural guess for a solution of (3.32) is that $\vec{J}_c$ is a null junction (defined in section 2). The dot product of a null junction with any other junction is zero when computed with respect to the metric on the junction lattice (see [3]). The same is not manifestly true after mapping with M_2 and computing the dot product with respect to the metric on $\mathcal{L}_C$. However we will argue below that this guess is correct for an appropriate choice of metric on $\mathcal{L}_C$. It is not be true for all choices of metric, since the element of $\mathcal{C}$ represented by a junction $\vec{J}_c$ depends on this choice, and is otherwise determined only up to an automorphism of $\mathcal{C}$. The proper null junctions, since they are proper, are mapped by M_2 to the identity element of $\mathcal{C}$. To get non-trivial elements of $\mathcal{C}$ we shall have to consider improper null junctions which are fractions of the proper null junctions. The subgroup $\mathcal{H}$ of $\mathcal{C}$ which acts trivially on all physical states is then given by

$$\mathcal{H} = \mathcal{N}/\mathcal{N}_{\mathcal{P}}\,, \tag{3.33}$$

where $\mathcal{N}_{\mathcal{P}}$ is the lattice of proper null junctions and $\mathcal{N}$ is a lattice of null junctions generated by the fractions of proper null junctions with integer P_m and Q_m.

To show (3.33), it is more convenient to work with the maps $\tilde{M}$ (as in (3.17)). Since $\vec{J}_{\mathcal{P}} = \sum_m (p_m \vec{w}_{p,m} + q_m \vec{w}_{q,m} \cdots)$ is a proper junction, (3.32) may be re-written as

$$\exp\left(i \sum_m \tilde{M}_2(\vec{J}_{c,m}) \cdot \tilde{M}_2(p_m \vec{w}_{p,m} + q_m \vec{w}_{q,m}) \right) = 1 \tag{3.34}$$

(3.34) must hold for all p_m and q_m satisfying $\sum_m p_m = \sum_m q_m = 0$. The solutions of (3.34) are junctions $\vec{J}_c$ with the property that

$$\exp\left(i\tilde{M}_2(\vec{J}_{c,m}) \cdot \tilde{M}_2(\vec{w}_{p,m}) \right) = z_p \tag{3.35}$$

and

$$\exp\left(i\tilde{M}_2(\vec{J}_{c,m}) \cdot \tilde{M}_2(\vec{w}_{q,m}) \right) = z_q \tag{3.36}$$

for all m for which $\vec{w}_{p,m}$ or $\vec{w}_{q,m}$ are defined.[2] There is no sum on the index m in the above expressions and z_p and z_q are independent of the index m. Due to the independence of (3.35) and (3.36) on m, the center of the universal cover, $\mathcal{C} = \prod_m \mathbb{Z}_{(N_m)} \times \mathbb{Z}_{(M_m)}$, must have a trivially acting subgroup $\mathcal{H}$ of the form $\mathbb{Z}_l \times \mathbb{Z}_r$.

Given a choice of metric, it very easy to construct a lattice of junctions representing elements of the trivially acting subgroup $\mathcal{H}$ of $\mathcal{C}$. For instance, one can first construct the lattice Γ_P of junctions $\vec{J}_c = \sum_m (P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m})$ for which the quantities

$$\tilde{M}_2(P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}) \cdot \tilde{M}_2(\vec{w}_{p,m}) \tag{3.37}$$

and

$$\tilde{M}_2(P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}) \cdot \tilde{M}_2(\vec{w}_{q,m}) \tag{3.38}$$

[2] $\vec{w}_{p,m}$ or $\vec{w}_{q,m}$ are not both defined for I_n fibers, corresponding to collections of 7-branes with the same charges.

are independent of the index m, and equal to an integer multiple of 2π. This clearly satisfies (3.35) and (3.36). Junctions satisfying this condition are proper and represent the identity element of $\mathcal{C}$. Let us call the lattice of such junctions $\Gamma_{\mathcal{P}}$. By considering fractions of these junctions such that P_m and Q_m remain integer, one preserves the independence of (3.37) and (3.38) on the index m, but gets representatives of all elements of $\mathcal{C}$ which act trivially on physical states. These improper junctions generate a lattice Γ containing $\Gamma_{\mathcal{P}}$. In terms of these two lattices,

$$\pi^1(\tilde{G}) = \mathcal{H} = \frac{\Gamma}{\Gamma_{\mathcal{P}}}\,. \tag{3.39}$$

In some cases the lattice of null junctions $\mathcal{N}$ is of the type Γ described above, but not always. Nevertheless there exists a choice of metric such that the trivially acting components of $\mathcal{C}$ are represented by $\vec{J}_c$ in $\mathcal{N}$. In general, for the appropriate choice of metric, the elements $\vec{J}_c$ in $\mathcal{N}$ satisfy the weaker conditions (3.35) and (3.36).

The lattice of null junctions has a proper sublattice $\mathcal{N}_{\mathcal{P}}$ corresponding to the identity element of $\mathcal{C}$. Generators of the lattice of proper null junctions can be obtained from string loops encircling all the 7-branes, with charges $(1,0)$ or $(0,1)$ in the upper half-plane. Pulling the lower half of the loop through the 7-branes (as in figures 4, 8), so that the entire junction lives in the upper half plane, gives a junction of the form $\sum_m P_m \vec{w}_m + Q_m \vec{w}_m$. Since these junctions map to the identity element of $\mathcal{C}$ under $\tilde{M}_2$, (3.35) and (3.36) are trivially satisfied, with $z_p = z_q = 1$.

The full lattice of null junctions $\mathcal{N}$ is obtained by taking fractions of the junctions in $\mathcal{N}_{\mathcal{P}}$ such that P_m and Q_m remain integer. The resulting lattice has has dimension 2, as desired to get $\pi^1(\tilde{G})$ of the expected form $\mathbb{Z}_l \times \mathbb{Z}_r$. In appendix. we show that the fractional null junctions satisfy (3.35) and (3.36) for the appropriate choice of metric.

Note that the lattice of proper null junctions $\mathcal{N}_{\mathcal{P}}$ does not depend on the type of Kodaira singularities in the elliptic surface, which can be changed by moving 7-branes around within the string loop which defined the proper null junction. However the lattice $\mathcal{N}$ which includes the fractional null junctions does depend on the type of Kodaira singularities, since the allowed fractions with integer P_m and Q_m depend on the 7-branes which are within each collection labelled by m.

In summary:

$\pi^1(\tilde{G})$ *is the quotient of the lattice of null junctions* $\mathcal{N}$ *by the lattice of proper null junctions* $\mathcal{N}_{\mathcal{P}}$.

$$\pi^1(\tilde{G}) = \mathcal{H} = \frac{\mathcal{N}}{\mathcal{N}_{\mathcal{P}}}\,. \tag{3.40}$$

We wish to emphasize that the null junctions determine not only $\pi^1(\tilde{G})$, but also the entire global structure of $\tilde{G}$. To obtain the full global structure of $\tilde{G}$, one requires the elements of the center of the universal cover of $\tilde{G}$ which act trivially in all representations of $\tilde{G}$. These elements are the image of the null junctions $\mathcal{N}$ under the map M_2.

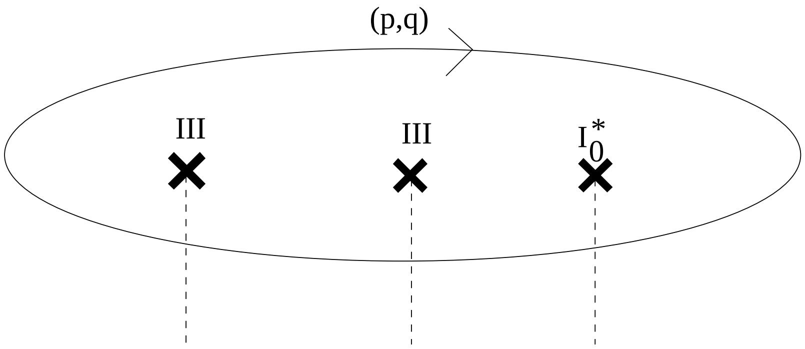

Figure 5: proper null junctions of a rational elliptic surface.

3.4 Examples for rational elliptic surfaces

To clarify some of the above statements, we illustrate them for some particular examples. Consider a rational elliptic surface with Kodaira fibers III, III, and I_0^*. The universal cover of $\tilde{G}$ is $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{Spin}(8)$, with center $\mathcal{C}=\mathbb{Z}_2\times\mathbb{Z}_2\times(\mathbb{Z}_2\times\mathbb{Z}_2)$. Arranging these fibers from left to right on the real axis, as in figure 5, we shall take $m=1,2$ to correspond to the two III fibers and $m=3$ to correspond to the I_0^* fiber. In terms of its decomposition to I_1's (individual 7-branes) this configuration is $(AAC)(AAC)(AAAABC)$ - see table 1.

Let us first consider $m=1$. Then, from table 2,

$$\begin{aligned}\vec{w}_{p,1} &= \left(\frac{1}{2},\frac{1}{2},0\right)\\ \vec{w}_{q,1} &= \left(\frac{1}{2},\frac{1}{2},-1\right).\end{aligned}\tag{3.41}$$

The lattice spanned by these junctions becomes $\mathbb{Z}_2$ (the center of $\mathrm{SU}(2)$) upon quotienting by the proper junctions. Furthermore

$$M_2(\vec{w}_{p,1})=M_2(\vec{w}_{q,1})=1 \bmod 2\,.\tag{3.42}$$

The junction $P_1\vec{w}_{p,1}+Q_1\vec{w}_{q,1}$ represents an element of the center determined by the map M_2. This element acts as follows in a representation containing a proper junction with asymptotic charges p_1,q_1;

$$\begin{aligned}\Psi_{\vec{J}_{\mathcal{P}}} &\to \exp\left(i\tilde{M}_2(P_1\vec{w}_{p,1}+Q_q\vec{w}_{q,1})\cdot\tilde{M}_2(p_1\vec{w}_{p,1}+q_1\vec{w}_{q,1})\right)\Psi_{\vec{J}_{\mathcal{P}}}=\\ &=(-1)^{(P_1+Q_1)(p_1+q_1)}\Psi_{\vec{J}_{\mathcal{P}}}\end{aligned}\tag{3.43}$$

For $m=3$ one has

$$\begin{aligned}\vec{w}_{p,3} &= \left(0,0,0,0,\frac{1}{2},\frac{1}{2}\right)\\ \vec{w}_{q,3} &= \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2},-\frac{1}{2}\right).\end{aligned}\tag{3.44}$$

The lattice spanned by these junctions becomes $\mathbb{Z}_2 \times \mathbb{Z}_2$ (the center of Spin(8)) upon quotienting by the proper junctions. We shall represent an element of the center by a pair of integers (j, k), each defined modulo 2. Then

$$\begin{aligned} M_2(\vec{w}_p) &= (1,0) \\ M_2(\vec{w}_q) &= (0,1)\,. \end{aligned} \tag{3.45}$$

The junction $P_3\vec{w}_{p,3} + Q_3\vec{w}_{q,3}$ represents an element of the center determined by the map M_2. In a representation containing a proper junction $\vec{J}_{\mathcal{P}}$ with asymptotic charges p_3 and q_3, this element of the center acts as follows;

$$\Psi_{\vec{J}_{\mathcal{P}}} \to (-1)^{P_3 p_3}(-1)^{Q_3 q_3}\Psi_{\vec{J}_{\mathcal{P}}}\,. \tag{3.46}$$

The lattice of proper null junctions is generated by

$$\vec{N}_1 = -\vec{w}_{p,1} - \vec{w}_{q,1} - \vec{w}_{p,2} + \vec{w}_{q,2} + 2\vec{w}_{p,3} \tag{3.47}$$

and

$$\vec{N}_2 = \vec{w}_{p,1} - \vec{w}_{q,1} - \vec{w}_{p,2} - \vec{w}_{q,2} + 2\vec{w}_{q,3} \tag{3.48}$$

corresponding to the string loops in figure 5 with $(p,q) = (1,0)$ and $(0,1)$ respectively. (3.47) and (3.48) are obtained upon pulling the lower half of the loop through the 7-branes and into the upper half plane (see figures 4, 8). Modulo proper null junctions, there is precisely one fractional null junction with integer P_i and Q_i. It is given by

$$\frac{1}{2}(\vec{N}_1 + \vec{N}_2) = -\vec{w}_{q,1} - \vec{w}_{p,2} + \vec{w}_{p,3} + \vec{w}_{q,3}\,. \tag{3.49}$$

This corresponds to the element $-1 \times -1 \times (-1 \times -1)$ in $\mathcal{C} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ In a representation containing a *proper* junction of the form

$$\vec{J}_{\mathcal{P}} = \sum_{i=1}^{3}(p_i\vec{w}_{p,i} + q_i\vec{w}_{q,i} + a_{k,i}\vec{w}_{k,i})\,, \tag{3.50}$$

this element of the center acts as follows;

$$\Psi_{\vec{J}_{\mathcal{P}}} \to (-1)^{(q_1+p_1)}(-1)^{(p_2+q_2)}(-1)^{p_3}(-1)^{q_3}\Psi_{\vec{J}_{\mathcal{P}}}\,. \tag{3.51}$$

This action is trivial when $\sum_m p_m = \sum_m q_m = 0$. Thus there is a $\mathbb{Z}_2$ subgroup of the center of the universal cover which acts trivially when $\sum_i p_i = \sum_i q_i = 0$. The semi-simple part of the gauge group is then

$$\tilde{G} = \frac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{Spin}(8)}{\mathbb{Z}_2}\,, \tag{3.52}$$

where the $\mathbb{Z}_2$ action is given by (3.51).

Let us briefly consider another example. Consider the rational elliptic surface with an I_3, I_1, and E_6 singularity. The universal cover is $\mathrm{SU}(3) \times E_6$ with center $\mathbb{Z}_3 \times \mathbb{Z}_3$. One can arrange these singularities from left to right on the real axis in the base, with indices $m = 1, 2$ and 3 labelling the I_3, I_1 and E_6 fibers respectively. The monodromies are (up to an overall $\mathrm{SL}(2, \mathbb{Z})$ conjugation)

$$
\begin{aligned}
M_1 &= \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & -0 \\ 1 & 1 \end{pmatrix}, \\
M_3 &= \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}.
\end{aligned} \tag{3.53}
$$

Note that this implies $q_1 = p_2 = 0$. It is easy to show (see the appendix) that $\vec{w}_{q,2}$ and $\vec{w}_{q,3}$ are proper, while $3\vec{w}_{p,1}$ and $3\vec{w}_{p,3}$ are minimal proper junctions, meaning that they can not be divided by any integer > 1 and remain proper. This accounts for the fact that the center of the universal cover is $\mathbb{Z}_3 \times \mathbb{Z}_3$. We choose a metric such that

$$
\begin{aligned}
\tilde{M}_2(\vec{w}_{p,1}) \cdot \tilde{M}_2(\vec{w}_{p,1}) &= \frac{2\pi}{3} \\
\tilde{M}_2(\vec{w}_{p.3}) \cdot \tilde{M}_2(\vec{w}_{p,3}) &= \frac{2\pi}{3}.
\end{aligned} \tag{3.54}
$$

Since the junctions $\vec{w}_{q,2}$ and $\vec{w}_{q,3}$ are proper, all dot products involving them are are integer multiples of 2π. The proper null junctions are

$$
\vec{N}_{(\tilde{p},\tilde{q})} = 3\tilde{q}\vec{w}_{p,1} + (3\tilde{q} - \tilde{p})\vec{w}_{q,2} - 3\tilde{q}\vec{w}_{p,3} + (\tilde{p} - 3\tilde{q})\vec{w}_{q,3}\,, \tag{3.55}
$$

where $(\tilde{p}, \tilde{q})$ indicates the charge in the upper half plane of the corresponding string loop surrounding all three singular fibers. The quotient lattice $\mathcal{N}/\mathcal{N}_{\mathcal{P}}$ is $\mathbb{Z}_3$ and is generated by the fractional null junction

$$
\vec{N} = \frac{\vec{N}_{(0,1)}}{3} = \vec{w}_{p,1} + \vec{w}_{q,2} - \vec{w}_{p,3} - \vec{w}_{q,3}\,. \tag{3.56}
$$

In a representation containing the proper junction $\vec{J}_{\mathcal{P}} = \sum_m p_m \vec{w}_{p,m} + q_m \vec{w}_{q,m} + \cdots$, the element of the center associated with (3.56) acts as follows;

$$
\begin{aligned}
\Psi_{\vec{J}_{\mathcal{P}}} &\to \exp\left(i\tilde{M}_2(\vec{N}) \cdot \tilde{M}_2(\sum_m p_m \vec{w}_{p,m} + q_m \vec{w}_{q,m}) \right) \Psi_{\vec{J}_{\mathcal{P}}} = \\
&= \exp\left(\frac{2\pi i}{3}(p_1 + p_3) \right) \Psi_{\vec{J}_{\mathcal{P}}}\,.
\end{aligned} \tag{3.57}
$$

Since $p_2 = 0$, we must have $p_1 + p_3 = 0$, and this action is trivial. Thus we find

$$
\tilde{G} = \frac{\mathrm{SU}(3) \times E_6}{\mathbb{Z}_3}\,, \tag{3.58}
$$

where the $\mathbb{Z}_3$ action is given by (3.57).

3.5 Gauge group for an elliptic K3.

In [1] the gauge group for F-theory compactifications on a $K3$ was computed at the boundary of moduli space at which a $K3$ degenerates to two rational elliptic surfaces intersecting over an elliptic curve. The spectrum in this case is consistent with a gauge group which is a direct product of groups computed for each rational elliptic surface. The means of computing $\pi^1(\tilde{G})$ in [1] differs from ours, as it involves identifying $\pi^1(\tilde{G})$ with the torsion component of the Mordell-Weil lattice $T(\Phi)$. Since we have identified $\pi^1(\tilde{G})$ with $\mathcal{N}/\mathcal{N}_{\mathcal{P}}$, our results explains the equivalence of $\mathcal{N}/\mathcal{N}_{\mathcal{P}}$ with $T(\Phi)$ which was observed for rational elliptic surfaces in [2]. Our methods are not restricted to rational elliptic surfaces however. They are equally applicable to generic elliptic K3's and, as we shall later argue, to some elliptically fibered three-folds.

The computation of the gauge group in [1] considers K3's which have degenerated to a pair of rational elliptic surfaces $\mathcal{R}_i$, and does not take into account certain multiplets which become infinitely massive in this limit. In this limit, the P^1 base of the $K3$ pinches to two P^1's meeting at a point. The spectrum considered in [1] consists of elements of $H_2(X,\mathbb{Z})$ within each rational elliptic surface. These elements of $H_2(X,\mathbb{Z})$ project to strings within each P^1 but not stretching between them. Strings stretching between the P^1 bases correspond to elements of $H_2(K3,\mathbb{Z})$ but not of $H_2(\mathcal{R}_i,\mathbb{Z})$. These strings are necessary to account for the difference between the second Betti number of a K3, $b_2 = 22$, and twice that of a rational elliptic surface, which has $b_2 = 10$. In the dual heterotic description on T^2, they correspond to strings winding cycles of the T^2 whose radii are becoming infinite in the degenerate limit. (see [15] for a discussion of string junctions and heterotic-IIB duality). The algebra associated with strings junctions on a rational elliptic surface is $\hat{E}_9$ (see [6]), whereas the Narain lattice of the heterotic theory on a torus is equivalent to the root lattice of $E_{10} \oplus E_{10}$. If one deforms the K3 away from the stable degeneration to a pair of rational elliptic surfaces, one can not neglect strings stretching between the two P^1 bases, as they are no longer infinitely massive.

Near the degeneration to an intersecting pair of rational elliptic surfaces $\mathcal{R}_i$, the 7-branes on the P^1 base of a K3 may be split into two groups of 12 with unit total monodromy. These groups of 7-branes live in the base of each $\mathcal{R}_i$ in the degenerate limit. The proper null junctions of a generic K3 are string loops surrounding *all* 24 7-branes. $\pi^1(\tilde{G})$ is generated by fractions of these null junctions having integer P_m and Q_m. The gauge group has the form $(G_1 \times G_2)/\Gamma$ with G_1 and G_2 simply connected and $\Gamma \equiv \mathcal{N}/\mathcal{N}_{\mathcal{P}}$, but is generically not a direct product of the form $G_1/\Gamma_1 \times G_2/\Gamma_2$ as found in [1]. The Γ_i computed in [1] are isomorphic to $\mathcal{N}_i/\mathcal{N}_{\mathcal{P},i}$, where $\mathcal{N}_i$ are null junctions in the base of $\mathcal{R}_i$. These junctions are related to string loops surrounding the relevant set of 12 7-branes. The elements in the center of the universal cover represented by junctions in $\mathcal{N}_i$ do not act trivially on junctions stretching between

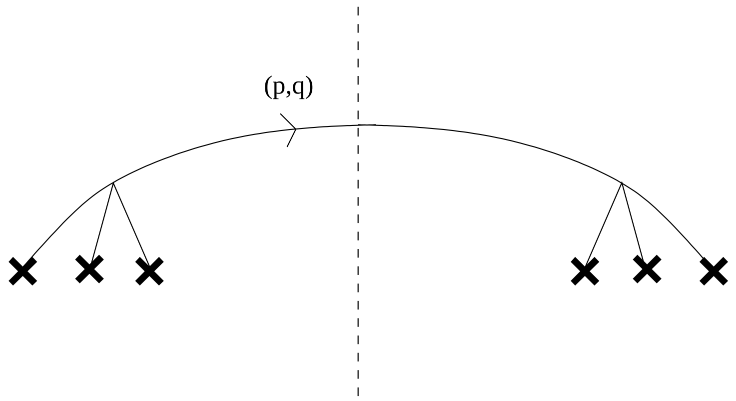

Figure 6: A string junction stretched between two sets of 12 7-branes in the base of an elliptic K3. In the limit in which the K3 degenerates to two rational elliptic surfaces, the dashed line (an S^1 equator on P^1) separating the two sets of 7-branes shrinks to a point.

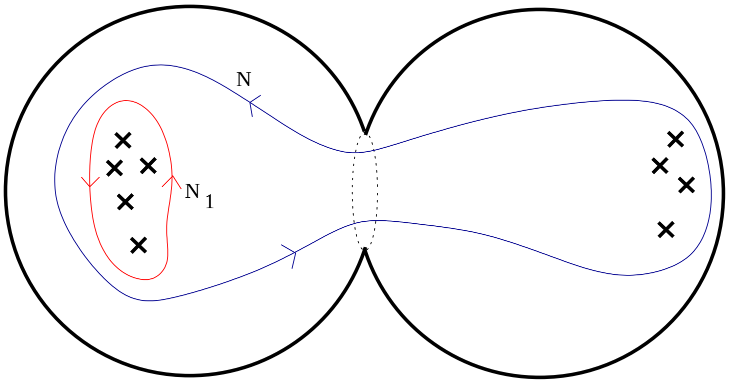

Figure 7: Pinching of the base of an elliptic K3 near the stable degeneration to a pair of intersecting rational elliptic surfaces. N_1 is equivalent to the 0 junction on the base of a rational elliptic surface, but not on the base of K3. N is equivalent to the 0 junction on the base of the K3, and its fractions determine $\pi^1(\tilde{G})$.

the two groups of 12 7-branes, since such junctions have a non-zero asymptotic (p, q) charges within each of the P^1 components of the base (see figure 6). Recall that the condition for null junctions to represent trivially acting elements in the center of the universal cover requires that the asymptotic charges of all states be zero. Note also that on a K3, string loops encircling just 12 of the branes (with unit total monodromy) are not really null (see figure 7), since they are inequivalent to the 0 string junction and represent a a nontrivial element of $H_2(K3, \mathbb{Z})$. In conclusion, for a $K3$

$\pi^1(G) = \mathcal{N}/\mathcal{N}_\mathcal{P}$ *where* $\mathcal{N}_\mathcal{P}$ *are null junctions arising from string loops surrounding all* 24 *7-branes.*

In the next section we compute the global structure of the gauge group for a K3, including all the U(1) factors as well.

4. The global structure including U(1) factors.

Thus far we have only considered the global structure of the semisimple part of the gauge group $\tilde{G}$. One can readily extend the discussion to include the U(1) factors. Consider a general junction characterized by Dynkin labels $a_{i,m}$ and charges p_m and q_m with $m = 1, \ldots, r$. The U(1) charges are related to the p_m and q_m charges. Since we require $\sum_m p_m = \sum_m q_m = 0.$, there are $2r - 2$ independent p_m and q_m charges. Of these, only $2r - 4$ are conserved. This is because there is dimension 2 lattice of null junctions, which are topologically equivalent to the zero junction. For a K3, the rank of the entire gauge group, including both $\tilde{G}$ and the U(1) factors, is 20, which is what one expects from duality with the heterotic string on T^2.

One can write the $2r - 4$ conserved U(1) charges $\mathcal{Q}_s$ as linear combinations of the independent p_m and q_m which are invariant under the additions of null junctions. The U(1) factors in the gauge group act as follows in representations containing junctions $\vec{J}$ with charges $\mathcal{Q}_s$;

$$\Psi_{\vec{J}} \to \exp(i\theta_s Q_s)\Psi_{\vec{J}}\,. \tag{4.1}$$

For certain values of θ_s, this may be the same as the action of some element in the center of $\tilde{G}$. The results of the previous section give the action of the center of $\tilde{G}$, so it is easy to compute the global structure when one includes all the U(1) factors.

As a simple example, consider a $K3$ with four I_0^* fibers. Locally, the gauge group is $\mathrm{U}(1)^4 \times \mathrm{Spin}(8)^4$. The lattice of proper null junctions is generated by

$$\vec{N}_1 = 2\vec{w}_{p,1} - 2\vec{w}_{p,2} + 2\vec{w}_{p,3} - 2\vec{w}_{p,4} \tag{4.2}$$

and

$$\vec{N}_2 = 2\vec{w}_{q,1} - 2\vec{w}_{q,2} + 2\vec{w}_{q,3} - 2\vec{w}_{q,4}\,. \tag{4.3}$$

Therefore the conserved U(1) charges are

$$\begin{aligned} Q_1 &= p_1 + p_2 = -p_3 - p_4 \\ Q_2 &= p_1 + p_4 = -p_2 - p_3 \\ Q_3 &= q_1 + q_2 = -q_3 - q_4 \\ Q_4 &= q_1 + q_4 = -q_2 - q_3\,. \end{aligned} \tag{4.4}$$

The center of $\mathrm{Spin}(8)^4$ is $(\mathbb{Z}_2 \times \mathbb{Z}_2)^4$. Using the results of the previous section, the action of the center in a representation containing the proper junction $\vec{J}_{\mathcal{P}} = \sum_m (p_m\vec{w}_{p,m} + q_m\vec{w}_{q,m} + a_{i,m}\vec{w}_{i,m})$ is;

$$\Psi_{\vec{J}_{\mathcal{P}}} \to \exp\left(i\pi \sum_m (l_m p_m + r_m q_m)\right)\Psi_{\vec{J}_{\mathcal{P}}}\,, \tag{4.5}$$

where for each m, l_m and r_m are integers indicating an element of $\mathbb{Z}_2 \times \mathbb{Z}_2$. There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $(\mathbb{Z}_2 \times \mathbb{Z}_2)^4$ which acts trivially on all $\vec{J}_{\mathcal{P}}$ with $\sum_m p_m = \sum_m q_m = 0$. This subgroup is generated by $(r_m, l_m) = (1, 0)$ for all m and $(r_m, l_m) = (0, 1)$ for all m. The semisimple part of the gauge group is therefore

$$\tilde{G} = \frac{\mathrm{Spin}(8)^4}{\mathbb{Z}_2 \times \mathbb{Z}_2}\,. \tag{4.6}$$

Among the *non*-trivially acting elements of $(\mathbb{Z}_2 \times \mathbb{Z}_2)^4$ are some which are equivalent to elements in the U(1) parts of the gauge group;

$$\exp(i\theta_i Q_i) = \exp\left(i\pi \sum_m (l_m p_m + r_m q_m)\right). \tag{4.7}$$

A U(1) factor with $\theta_i = \pi$ for any i is equivalent to a non-trivially acting element in the center of $\mathrm{Spin}(8)^4$. Thus we find that the full gauge group is

$$G = \frac{\mathrm{U}(1)^4 \times \mathrm{Spin}(8)^4}{\mathbb{Z}_2^6}\,. \tag{4.8}$$

5. Elliptically fibered Calabi-Yau three-folds and $\pi^1(G)$.

String junction methods may also be applicable to the determination of the global structure of the gauge group for F-theory compactifications on elliptic Calabi-Yau three-folds. Some new complexities arise in this case due to singularities of the discriminant curve over which different Kodaira types collide. The relation of $\pi^1(\tilde{G})$ to the torsion component of the Mordell-Weil lattice of an elliptic Calabi-Yau three-fold was shown to hold in certain special instances in [1]. However the Mordell-Weil lattices of elliptic Calabi-Yau three-folds have not been generally classified, and the relation of its torsion component to $\pi^1(\tilde{G})$ is un-proven. Our purpose in this section is to suggest a general approach to obtain the global structure of the gauge group using string junctions.

Consider a K3 fibered Calabi-Yau three-fold, of "perturbative" type, for which the K3 fibers become more singular at isolated points P in the P^1 base due to the collision of Kodaira fibers. In this case we propose to determine the global structure of the gauge group as follows. One first computes the group G associated to a generic K3 fiber using the methods of the previous sections. The actual gauge symmetry *algebra* $\mathcal{G}$ may be a sub-algebra of that of G, and in some instances is non-simply laced [20, 19, 22]. We conjecture that the gauge symmetry group is simply a subgroup of G with the algebra $\mathcal{G}$. This will be true provided that the string junction lattice of the Calabi-Yau is that of a generic $K3$, even if the gauge symmetry algebra is a sub-algebra of that of a generic $K3$. This requires that junctions corresponding to vector multiplets in compactifications on a K3 X do not disappear from the spectrum in compactification on a Calabi-Yau Y for which X is the generic fiber, but may instead correspond to hypermultiplets.

There is good evidence that this is the case. For instance, massless vector multiplets arise from membranes wrapping vanishing cycles in $H_2(Y, \mathbb{Z})$, whereas massless hypermultiplets arise from membranes wrapping unions of rational curves [22] in the blowup of the elliptic fiber over the singular points where components of the discriminant curve collide. These unions of rational curves do not necessarily correspond to elements of $H_2(Y, \mathbb{Z})$. A discussion of hypermultiplets which are not contained in $H_2(Y, \mathbb{Z})$ appeared originally in [21]. In the non-simply laced cases, there is a monodromy which acts on the Dynkin diagram associated to X [20, 19, 22]. This monodromy removes some roots from the gauge symmetry algebra, however there are hypermultiplets arising from unions of rational curves in the elliptic fiber which are exchanged under the monodromy.

The string junction lattice for a Calabi-Yau three-fold is currently being investigated in (see [16]). String junctions on an elliptically fibered three-fold differs somewhat from those on an elliptic surface (a two-fold), since the string junctions live in a base of four real dimensions rather than two. The discriminant locus is a collection of complex curves, rather than points. Strings may end on these curves, and in addition to the equivalence relations between strings related by smooth deformations and Hanany-Witten transitions, there is an additional equivalence between strings related by sliding endpoints along the discriminant curve. A subtlety arises due to the singular points of the discriminant curve, where different smooth components meet or intersect. A well defined string junction lattice can be obtained by finding the equivalence classes of string junctions in the space $B - s_\alpha$ where B is the base of the elliptically fibered Calabi-Yau and s_α are the singular points of the discriminant curve at which Kodaira types collide.

Consider again a K3-fibered Calabi-Yau X, for which the K3 is fibered over a lower P^1 and the $K3$ fiber is itself an elliptic fibration over an upper P^1. We assume the $K3$ fiber degenerates at isolated points $\mathcal{Q}_s$ in the lower P^1, over which smooth components of the discriminant curve collide. Via an equivalence relation, any string junction can be brought entirely within an upper P^1, over a point $\mathcal{Q}$ in the lower P^1. This suggests that the string junction lattice is that of a generic $K3$ fiber, up to a possible quotient by the action of a monodromy as $\mathcal{Q}$ encircles a point $\mathcal{Q}_s$ at which the $K3$ becomes more singular due to the collision of Kodaira fibers. Within a generic upper P^1, string junctions end at points $\mathcal{Y}_i$ where the discriminant curve intersects the upper P^1. As Q encircles Q_s, the $\mathcal{Y}_i$ corresponding to Kodaira fibers which collide at $\mathcal{Q} = \mathcal{Q}_s$ move around each other, sweeping out a braid representation of a knot or link. This generates a natural action of the braid group on string junctions;

$$\vec{J} \to \hat{B}\vec{J}, \tag{5.1}$$

where $\hat{B}$ is an element of the braid group, and $\vec{J}$ is a string junction defined over the point $\mathcal{Q}$. The junctions $\vec{J}$ and $\hat{B}\vec{J}$ are equivalent in the three-fold, even if they are inequivalent when restricted to a point $\mathcal{Q}$. If $\hat{B}$ acts non-trivially, then any junction is equivalent to one with endpoints on a subset of the points $\mathcal{Y}_i$ One expects $\hat{B}$ to act non-trivially if the braid is non-minimal, i.e. if there is another braid representation of the same knot with a fewer number of strands.

Given some of the above statements about the hypermultiplet spectrum for the Calabi-Yau compactifications, one would expect that the the braid which arises when $\mathcal{Q}$ encircles $\mathcal{Q}_s$ is always minimal so that the quotient acts trivially, although we will not be able to prove this here. Related questions concerning the properties of algebraic knots are discussed in [16].

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A. Properties of fractional null junctions

We will show below that the null junctions with integer P_m and Q_m obtained from fractions of proper null junctions satisfy (3.35) and (3.36).

Consider a proper null junction obtained by starting with a string loop surrounding all 7-branes and deforming it into the upper half plane by pulling the lower part of the loop through all the 7-branes. Suppose also that the charge of the string loop is $(\tilde{p}, \tilde{q})$ in the upper half plane. The resulting junction $\vec{N}_{(\tilde{p},\tilde{q})}$ has the form

$$\vec{N}_{(\tilde{p},\tilde{q})} = \sum_m \vec{J}_{c,m} = P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m} \,. \tag{A.1}$$

The full (one dimensional) lattice of null junctions proportional to $\vec{N}_{(\tilde{p},\tilde{q})}$ is generated by

$$\vec{N}_{(\tilde{p},\tilde{q})}/L_{\tilde{p},\tilde{q}} \,, \tag{A.2}$$

where

$$L_{\tilde{p},\tilde{q}} = \gcd(P_1, P_2, \ldots, Q_1, Q_2, \ldots) \tag{A.3}$$

and gcd indicates the greatest common divisor. We wish to show that (3.35) and (3.36) are satisfied by the null junctions in this lattice. In other words, we wish to show that

$$\exp\left(i \frac{1}{L_{\tilde{p},\tilde{q}}} \tilde{M}_2(P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}) \cdot \tilde{M}_2(\vec{w}_{p,m}) \right) = z_p \tag{A.4}$$

and

$$\exp\left(i \frac{1}{L_{\tilde{p},\tilde{q}}} \tilde{M}_2(\vec{w}_{p,m} + Q_m \vec{w}_{q,m}) \cdot \tilde{M}_2(\vec{w}_{q,m}) \right) = z_q \,, \tag{A.5}$$

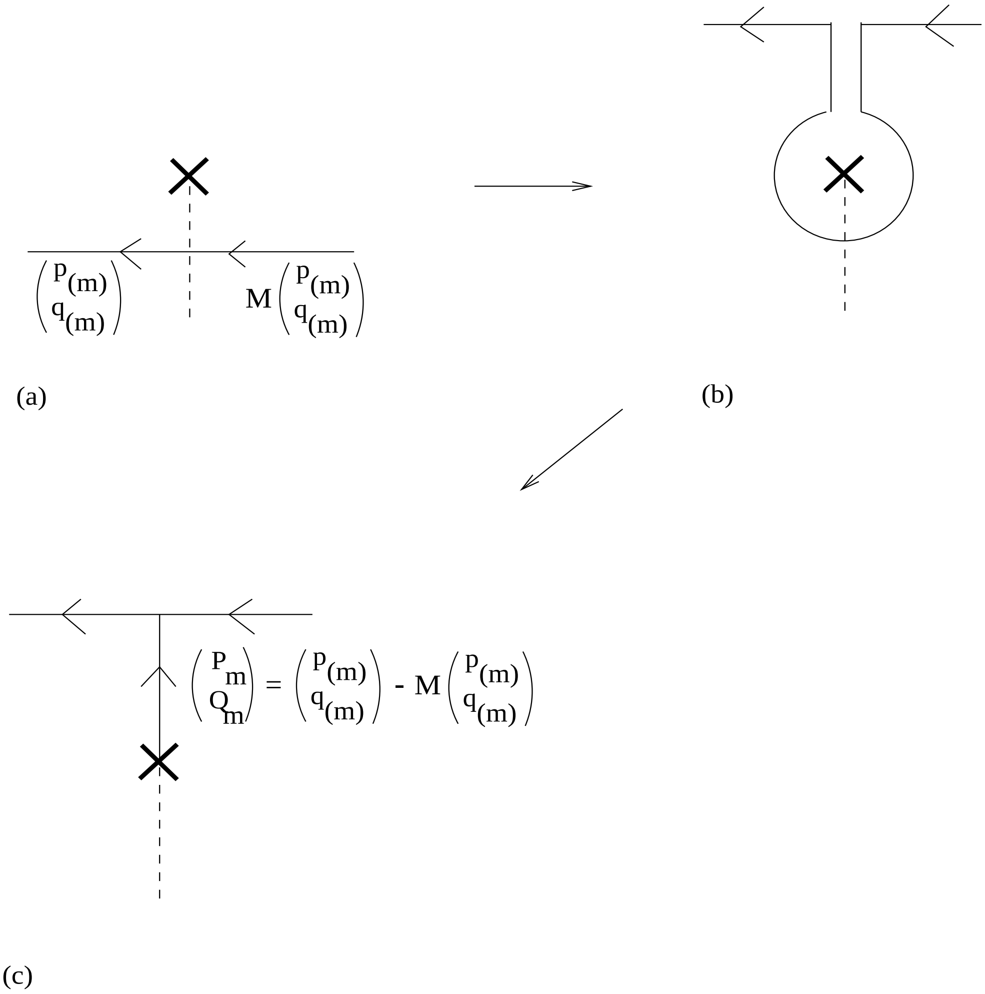

Figure 8: Hanany-Witten transition upon pulling a segment of a string loop through a collection of 7-branes.

where there is no sum on the index m in the above expressions and z_p and z_q are independent of the index m. It will suffice to show (A.4) and (A.5) for a generating pair of $\vec{N}_{(\tilde{p},\tilde{q})}$ with $\gcd(\tilde{p},\tilde{q}) = 1$, such as $(1,0)$ and $(1,0)$,.

Since the junction $\vec{J}_{c,m} = P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}$ (with no sum on m) is proper,

$$\tilde{M}_2(P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}) \cdot \tilde{M}_2(\vec{w}_{p,m}) = 2\pi j_m \tag{A.6}$$

and

$$\tilde{M}_2(P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}) \cdot \tilde{M}_2(\vec{w}_{q,m}) = 2\pi k_m \tag{A.7}$$

for integer j_m and k_m. In the absence of constraints on these integers (A.4) and (A.5) would not be satisfied. Fortunately, there are constraints. To see this let us consider the effect of pulling a segment of a string loop below the m'th collection of 7-branes into the upper half plane. The charges of this string segment are $(p_{(m-1)}, q_{(m-1)})$ to the left of the branch cut and $(p_{(m)}, q_{(m)})$ to the right of the branch cut, (see figure 8), where these are related by the $\mathrm{SL}(2,\mathbb{Z})$ monodromy M_m,

$$\begin{pmatrix} p_{(m+1)} \\ q_{(m+1)} \end{pmatrix} = M_m \begin{pmatrix} p_{(m)} \\ q_{(m)} \end{pmatrix}. \tag{A.8}$$

Upon pulling this segment through the 7-branes one finds (see figure 8),

$$\begin{pmatrix} P_m \\ Q_m \end{pmatrix} = (I - M_m) \begin{pmatrix} p_{(m)} \\ q_{(m)} \end{pmatrix}. \tag{A.9}$$

We will first consider the case in which none of the monodromies around any of the collections of coincident 7-branes has an eigenvalue equal to one. This excludes the I_n series. In this case the matrix $I - M_m$ is invertible for all m. It follows that the junction $J_{c,m} = P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}$ (with no sum on m) is a "minimal" proper junction, in the sense that it can not be divided by any integer greater than 1 and remain proper. If it could, then a proper junction would be related by a Hanany-Witten transition to a segment of string junction with fractional charges, i.e. a fraction of $(p_{(m-1)}, q_{(m-1)})$ and $(p_{(m)}, q_{(m)})$ to the left and right of the branch cut respectively, where $\gcd(p_{(m-1)}, q_{(m-1)}) = \gcd(p_{(m)}, q_{(m)}) = 1$. However Hanany-Witten transitions do not relate proper and fractional strings.

The fact that $\vec{J}_{c,m} = P_m \vec{w}_{p,m} + Q_m \vec{w}_{q,m}$ is, for each m, a minimal proper junction is the essential fact that allows one to choose a metric for which (A.4) and (A.5) are satisfied. Consider what would happen if this were not the case. For instance suppose that $\gcd(P_1, P_2, \ldots, Q_1, Q_2, \ldots) = 2$ and $\vec{J}_{c,1}$ is minimal but $\vec{J}_{c,2}$ is non-minimal until divided by two. In this case $\vec{N}_{\tilde{p},\tilde{q}}/2$ does not satisfy (A.4) and (A.5). $\vec{J}_{c,2}/2$ would be proper and correspond to a trivial element of the center of G_2, but $\vec{J}_{c,1}/2$ would be improper and correspond to a non-trivial element of the center of G_1. In fact however, $\vec{J}_{c,m}$ is a minimal proper junction for all m, and one can choose a metric such that j_m and or k_m are either $+1$ for all m or 0 for all m. Note that it is never true that both j_m and k_m are zero for all m if $\gcd(P_1, P_2, \ldots, Q_1, Q_2, \ldots) > 1$, since the fractional null junctions are improper and represent non-trivial elements of the center of the universal cover.

We wish to emphasize that the correspondence between the fractional null junctions and the trivially acting elements in the center of the universal cover depends crucially on the choice of metric. For instance, the center of G_m has an automorphism (complex conjugation) which flips the sign of j_m or k_m. After acting with such an automorphism, the null junctions correspond to different elements in the center of the universal cover which may not act trivially on physical states.

We now consider the case in which there are I_n fibers. These correspond to collections of 7-branes all of which carry the same (p,q) charge. This case is somewhat more subtle, since $I - M_m$ is not invertible. For example, if the charge of the 7-branes is $(1,0)$, then a segment of string with charge $(1,0)$ passing under the 7-brane makes no contribution to the P_m and Q_m charge upon pulling the string into the upper half plane;

$$(I - M_m)\begin{pmatrix}1\\0\end{pmatrix} = 0\,. \tag{A.10}$$

The only contribution to P_m comes from the $q_{(m)}$ charge of the string segment below the I_n locus. Consequently, the junction $\vec{J}_{c,m}$ is not necessarily a minimal proper junction. It is minimal only after division by $q_{(m)}$.

Let us make a choice of a generating pair $\vec{N}_{\tilde{p},\tilde{q}}$ with $(\tilde{p},\tilde{q}) = (0,1)$ or $(1,1)$. The group associated with an I_n fiber is SU(n), with center $\mathbb{Z}_n$. If the vanishing cycle (7-brane charge) of this fiber is $(1,0)$, then the minimal proper junction is $n\vec{w}_{p,m}$. Upon pulling the lower segment of the string loop into the upper half plane, one finds

$$\vec{J}_{c,m} = q_{(m)} n \vec{w}_{p,m}\,. \tag{A.11}$$

We can choose a metric such that

$$\tilde{M}_2(\vec{J}_{c,m}) \cdot \tilde{M}_2(\vec{w}_{p,m}) = q_{(m)} n \tilde{M}_2(\vec{w}_{p,m}) \cdot \tilde{M}_2(\vec{w}_{p,m}) = 2\pi i q_{(m)}\,. \tag{A.12}$$

Then

$$\begin{aligned}\exp\left(i\frac{1}{\gcd(P_1,P_2,\ldots,Q_1,Q_2\ldots)}\tilde{M}_2(\vec{J}_{c,m})\cdot\tilde{M}_2(\vec{w}_{p,m})\right) &=\\ = \exp\left(2\pi i\frac{q_{(m)}}{\gcd(P_1,P_2,\ldots,Q_1,Q_2\ldots)}\right)&.\end{aligned} \tag{A.13}$$

Now

$$q_{(m)} = q_{(1)} + Q_1 + Q_2 + \cdots + Q_m = \tilde{q} + Q_1 + Q_2 + \cdots + Q_m\,, \tag{A.14}$$

so that

$$\begin{aligned}\exp\left(2\pi i\frac{q_{(m)}}{\gcd(P_1,P_2,\ldots,Q_1,Q_2\ldots)}\right) &= \exp\left(2\pi i\frac{\tilde{q}}{\gcd(P_1,P_2,\ldots,Q_1,Q_2\ldots)}\right)\\ &= \exp\left(2\pi i\frac{1}{\gcd(P_1,P_2,\ldots,Q_1,Q_2\ldots)}\right).\end{aligned}$$

This is precisely the same behavior as if $\vec{J}_{c,m}$ were a minimal null junction; there is no multiple of $\vec{J}_{c,m}/\gcd(P_1,P_2,\ldots,Q_1,Q_2\cdots)$ which is both proper and "smaller" than $\vec{J}_{c,m}$.

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